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A Class of Integrable Minisuperspaces

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Abstract

It is shown how the Wheeler-DeWitt and Hamilton-Jacobi equations for a two-dimensional minisuperspace may be solved in full generality if the super-potential of the wavefunction is a separable function of the minisuperspace null coordinates. In this case, the system may be viewed as a constrained oscillator-ghost-oscillator model. These solutions describe the quantum cosmology of a renormalizable two-dimensional dilaton gravity theory and the quantum dynamics of the event horizon in Rindler space-time.

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In the canonical quantization of General Relativity, the classical Hamiltonian constraint $\mathcal{H} = 0$ is viewed as a quantum mechanical operator that annihilates the physical states Ψ of the Universe: $\mathcal{H}\Psi = 0$. This functional differential equation governs the dynamics of the wavefunction in an infinite-dimensional configuration space known as superspace. Currently it is not known how to solve this equation in full generality, but progress can be made by imposing a high degree of symmetry on the system and considering a dimensional reduction of superspace to a finite-dimensional sector known as minisuperspace. If the minisuperspace is two-dimensional, the Hamiltonian constraint (Wheeler-DeWitt equation [1]) has the generic form of a hyperbolic, second-order partial differential equation:

$$\left[\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\beta^2} - 4m^2(\alpha,\beta) \right] \Psi = 0, \tag{1}$$

where the 'superpotential' $m^2(\alpha,\beta)$ is some function of the minisuperspace coordinates (α,β).

Equations of this form also arise in a number of other fundamental problems in physics and cosmology and it is therefore important to develop techniques that allow exact solutions to be derived in a straightforward manner. The purpose of this letter is to illustrate how this equation may be solved for a wide class of $m^2(\alpha,\beta)$. We assume that the superpotential is positive definite, although the analysis is easily extended to negative potentials. Examples include minisuperspaces corresponding to a $(1 + 1)$-dimensional dilaton gravity theory and the event horizon of the Rindler space-time.

If the function $m^2(\alpha,\beta)$ is independent of the minisuperspace coordinates, Eq. (1) may be transformed into the canonical form

$$\left[\frac{\partial^2}{\partial u \partial v} - 1 \right] \Psi = 0, \tag{2}$$

where $u = m(\alpha+\beta)$ and $v = m(\alpha-\beta)$ are null coordinates over minisuperspace. This equation has been studied previously by Page [2] and one family of solutions is given by $\Psi_b = e^{-ibu+iv/b}$, where b is an arbitrary complex constant. This family forms the basis for the general solution which can be expressed as the two-dimensional integral $\Psi_{\text{gen}} = \int d^2b L(b,b^*)\Psi_b$, where the density $L(b,b^*)$ is an arbitrary function. The wavefunction is bounded and square-integrable if $L(b,b^*)$ is finite and only supported in a compact region when $\text{Im} b < 0$ [2]. In this case, Cauchy's theorem implies that the two-dimensional integral may be replaced by the line integral

$$\Psi_{\text{gen}} = \int_{-\infty}^{+\infty} db M(b)\Psi_b, \tag{3}$$

where $M(b)$ is an arbitrary function.

In general, however, it is rather difficult to evaluate the integral (3) analytically. On the other hand, new exact solutions may be generated after further coordinate

transformations. For example, the transformation $x \equiv \sqrt{v} - \sqrt{u}$, $y \equiv \sqrt{v} + \sqrt{u}$ maps the system onto a constrained oscillator-ghost-oscillator model, where the Wheeler-DeWitt equation is the wave equation for two harmonic oscillators with equal and opposite energy [2, 3, 4]. The general solution is $\Psi = \sum_{n=0}^{\infty} c_n \Psi_n$, where c_n are arbitrary complex constants,

$$\Psi_n = (2^n n!)^{-1} H_n(x) H_n(y) \exp[-(x^2 + y^2)/2], \tag{4}$$

and H_n is the Hermite polynomial of order n. The ground state is defined by $n = 0$ and Ψ_n form a discrete basis for all bounded wavefunctions satisfying the unit-mass Klein-Gordon equation (2).

Thus, the general bounded and square-integrable solution to Eq. (1) may be found whenever this equation can be transformed into Eq. (2). We now determine the constraints on the functional form of the superpotential that must be satisfied for this to be possible. To proceed we introduce new variables $u = u(\sigma)$ and $v = v(\tau)$ that are *arbitrary* functions of the minisuperspace null coordinates $\sigma \equiv \alpha + \beta$ and $\tau \equiv \alpha - \beta$. These new variables satisfy the boundary conditions $\partial u / \partial \alpha = \partial u / \partial \beta$ and $\partial v / \partial \alpha = -\partial v / \partial \beta$ and these constraints ensure that the derivative terms in Eq. (1) are transformed into the canonical form:

$$\left[\frac{\partial u}{\partial \alpha} \frac{\partial v}{\partial \alpha} \frac{\partial^2}{\partial u \partial v} - m^2 \right] \Psi = 0. \tag{5}$$

It follows that Eq. (5) reduces to Eq. (2) if the new variables u and v are themselves solutions to the equation

$$m^2 = \frac{\partial u}{\partial \alpha} \frac{\partial v}{\partial \alpha} = \frac{du}{d\sigma} \frac{dv}{d\tau}. \tag{6}$$

Consequently, the general solution to Eq. (1) can be deduced *immediately* if a solution to the constraint equation (6) can be found. Effectively, the problem of solving the linear, second-order partial differential equation (1) has been reduced to finding a solution to the non-linear, *first-order* equation (6) and in many cases it is considerably easier to solve this latter equation.

Indeed, it is clear from the second equality in Eq. (6) that when the superpotential has the generic form

$$m^2(\alpha, \beta) = m_+(\sigma) m_-(\tau), \tag{7}$$

where $m_\pm$ are some known analytical functions, Eq. (6) admits the general separable solution

$$u = z \int^{\sigma} d\sigma' m_+(\sigma'), \qquad v = z^{-1} \int^{\tau} d\tau' m_-(\tau'), \tag{8}$$

where z is an arbitrary separation constant. Thus, the general solution to Eq. (1) is given by Eqs. (3) and (8) whenever the superpotential is a separable function of the minisuperspace null coordinates.

As well as being interested in exact solutions to the Wheeler-DeWitt equation, however, it is also important to determine semi-classical wavefunctions in the WKB approximation corresponding to $\hbar \to 0$. In this approximation the wavefunction is viewed as a linear superposition of waves $\Psi \sim e^{-iS/\hbar}$, where S is interpreted as the classical action that satisfies the Hamilton-Jacobi equation.

When Eq. (6) is satisfied the Hamilton-Jacobi equation has the form

$$\frac{\partial S}{\partial r}\frac{\partial S}{\partial v} = -r, \tag{9}$$

where $r \equiv \sqrt{2u}$ and this equation can be solved in full generality by employing a Legendre transformation [5]. We define new variables $\xi \equiv \partial S/\partial r$, $\eta \equiv \partial S/\partial v$ and a new function $\rho(\xi, \eta) \equiv r\xi + v\eta - S(r, v)$. Partial differentiation with respect to ξ implies that $r = \partial \rho/\partial \xi$ and substitution of this result into Eq. (9) implies that

$$\frac{\partial \rho}{\partial \xi} = -\xi\eta. \tag{10}$$

The *general* solution to this equation is

$$\rho(\xi, \eta) = -\frac{1}{2}\eta\xi^2 + f(\eta), \tag{11}$$

where $f(\eta)$ is an arbitrary function of η, and the calculation is completed by transforming back into the old variables. We deduce immediately that $r = -\xi\eta$ and partial differentiation of the solution (11) with respect to η implies that

$$\frac{\partial \rho}{\partial \eta} = v = -\frac{1}{2}\xi^2 + \frac{df(\eta)}{d\eta}. \tag{12}$$

Thus, it follows from the definition of ρ that

$$S = -\frac{2u}{\eta} - f(\eta) + \eta\frac{df(\eta)}{d\eta}$$
$$v = -\frac{u}{\eta^2} + \frac{df(\eta)}{d\eta}. \tag{13}$$

All developable solutions to Eq. (9) can be written in this fashion and Eq. (13) therefore represents the general solution to the Hamilton-Jacobi equation (9) in parametric form. In principle, once $f(\eta)$ is specified we can determine $\eta = \eta(u, v)$ from the second equation in (13) and substituting this result into the first equation yields the action in terms of u and v, or equivalently, in terms of the original minisuperspace coordinates α and β via Eq. (8) [6].

Some particular examples are of interest. One solution to Eq. (9) is $S = bu - b^{-1}v$ and this semi-classical wavefunction is identical to the family of exact solutions Ψ_b that satisfy the Wheeler-DeWitt equation (2). In this case the WKB approximation is exact. A second separable solution is given by $S = \pm 2i\sqrt{(u - u_i)(v - v_i)}$, where

$\{u_i, v_i\}$ are constants and this solution corresponds to $f(\eta) = -u_i/\eta$. Finally, exact expressions for $\eta(u, v)$ can also be found if $f \propto \eta$, $f \propto \eta^{\pm 3}$ and $f \propto \ln \eta$.

There are a number of interesting minisuperspaces for which the above techniques and solutions are relevant. In many cases the superpotential of the wavefunction is independent of one of the null coordinates, i.e. it is a single function of either σ or τ. This is the case for the minisuperspace corresponding to a dimensional reduction of five-dimensional Einstein gravity [7]. The Kantowski-Sachs and Bianchi III Universes containing vacuum energy also admit this type of separation [8]. More generally, if $m^2(\alpha, \beta) = \alpha - \beta$, Eq. (1) may be separated into Airy's equation and this implies that the general solution to Eq. (2) may also be expressed as a combination of Airy functions $\text{Ai}(\alpha + z)$ and $\text{Bi}(\beta + z)$, where z is an arbitrary constant.

The gravitational component of the Wheeler-DeWitt equation derived from a $(1 + 1)$-dimensional dilaton gravity theory leads to a superpotential of this form. Recently there has been considerable interest in $(1+1)$-dimensional theories of gravity coupled to matter [4, 9, 10, 11], since these are exact string theories if the space-time is interpreted as the string world sheet. In particular, the one-loop, string-inspired effective action discussed in Ref. [10] has the form

$$S = \frac{1}{\pi} \int_M d^2 x \left[-\frac{1}{\kappa} \partial_+ \chi \partial_- \chi + \frac{1}{\kappa} \partial_+ \Omega \partial_- \Omega \right.$$
$$\left. + \mu^2 e^{2(\chi - \Omega)/\kappa} + \frac{1}{2} \sum_{j=1}^{N} \partial_+ f_j \partial_- f_j \right] \tag{14}$$

in the conformal gauge, where M is the two-dimensional manifold, f_j are conformal scalar fields, Ω is a rescaled dilaton field and χ is a Liouville-type field. The constants $\kappa = (N - 24)/12$ and μ^2 are assumed to be positive-definite [12]. The Wheeler-DeWitt equation corresponding to this renormalizable model of dilaton gravity has been derived by calculating the Virasoro generators for a space-time topology $R \times S^1$ [13] and has the form

$$\left[\frac{\kappa}{4} \frac{\partial^2}{\partial \chi_0^2} - \frac{\kappa}{4} \frac{\partial^2}{\partial \Omega_0^2} - \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial f_{j0}^2} \right.$$
$$\left. - 4\mu^2 e^{2(\chi_0 - \Omega_0)/\kappa} - \kappa - 2 \right] \Psi = 0, \tag{15}$$

where χ_0, etc., represent the zero modes of the harmonic expansion of the fields on the cylinder.

If we separate the wavefunction into its gravitational and matter components with the ansatz $\Psi = \Phi(\chi_0, \Omega_0) \varphi(f_{j0})$, introduce a separation constant $Z^2 \leq \kappa + 2$ and identify (α, β) with (χ_0, Ω_0), it is readily seen that Φ statisfies Eq. (1) with a superpotential of the form

$$\kappa m^2 = 4\mu^2 e^{2(\chi_0 - \Omega_0)/\kappa} + (\kappa + 2 - Z^2). \tag{16}$$

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Since this is a single function of $(\chi_0 - \Omega_0)$, Eq. (8) implies that the solution may be written in terms of the new variables

$$u = z(\chi_0 + \Omega_0)$$
$$v = z^{-1}\left[2\mu^2 e^{2\tau_0/\kappa} + \kappa^{-1}(\kappa + 2 - Z^2)\tau_0\right],$$

(17)

where $\tau_0 = \chi_0 - \Omega_0$.

A third class that has many applications is

$$m^2 = m_0^2 e^{\epsilon\alpha + \gamma\beta},$$

(18)

where $\{m_0, \gamma, \epsilon\}$ are arbitrary constants subject to the single restriction that $\gamma \neq \pm\epsilon$ [14]. Eq. (18) separates into $m_+(\sigma) = m_0 e^{(\epsilon+\gamma)\sigma/2}$ and $m_-(\tau) = m_0 e^{(\epsilon-\gamma)\tau/2}$ and this class may be solved with the coordinate transformation

$$u = \frac{2zm_0}{\epsilon + \gamma} e^{(\epsilon+\gamma)\sigma/2}, \qquad v = \frac{2m_0}{z(\epsilon - \gamma)} e^{(\epsilon-\gamma)\tau/2}.$$

(19)

Spatially flat, isotropic Universes containing a single scalar field that self-interacts through an exponential potential lead to a superpotential of this form and many natural extensions to Einstein gravity, including some Kaluza-Klein and R^n higher-order gravity theories, are conformally equivalent to this theory. Superpotentials of this type also arise in some Bianchi class A Universes [15].

Moreover, this class may be relevant to the study of quantum black holes. A distant, static observer can describe a quasi-stationary, classical black hole in terms of a 'stretched horizon' or 'membrane' located just outside the event horizon [16]. However, since this membrane is not seen by a freely-falling observer, the physical reality of such an object has remained uncertain. In principle this ambiguity is resolved by the hypothesis of black hole complementarity [17]; if the separate measurements made by the stationary and freely-falling observers are complementary, the latter observer has no way of reporting the non-existence of the membrane to the former and as far as the static observer is concerned the membrane is a real, physical quantity.

Motivated by these considerations, Maggiore recently proposed a quantum description of black holes by identifying the membrane's degrees of freedom as the variables to be quantized [18]. In this picture the membrane dynamics is determined by the Dirac action of a closed relativisitic bosonic membrane in four dimensions:

$$S_{\text{mem}} = -\rho \int_{\mathcal{K}} d^3\xi \sqrt{-\det\gamma_{ab}}$$
$$\gamma_{ab} = g_{\mu\nu}\frac{\partial x^\mu}{\partial\xi^a}\frac{\partial x^\nu}{\partial\xi^b},$$

(20)

where ρ is the membrane tension, γ_{ab} $(a,b = 0,1,2)$ is the metric induced on the membrane by the space-time metric $g_{\mu\nu}$ and $x^\mu = x^\mu(\xi^a)$ $(\mu,\nu = 0,1,2,3)$ are the embedding equations that map the membrane's manifold $\mathcal{K}$ into its world-volume in space-time [19]. The coordinates ξ^a parametrize this world-volume.

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As with quantum cosmology, however, it is necessary to impose a high degree of symmetry on the theory in order to proceed analytically. A very useful approximation to a black hole event horizon is given by the horizon of the Rindler space-time

$$ds^2 = -g^2 z^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{21}$$

This is the metric as seen by an observer undergoing uniform acceleration g in Minkowski space-time. It has a future (past) event horizon at $z = t$ ($z = -t$) that is formally equivalent to the Schwarzschild black hole event horizon in the limit of infinite mass.

In order to determine the quantum dynamics of this horizon, we assume that the embedding equations have the form $x^\mu = [x^0(\xi^0), \xi^1, \xi^2, z(\xi^0)]$. The non-zero components of γ_{ab} are then calculated from Eq. (20) and the action simplifies to

$$S_{\text{mem}} = -\rho' \int ds \left[G_{AB} \frac{dY^A}{ds} \frac{dY^B}{ds} \right]^{1/2}, \tag{22}$$

where $\rho' \equiv \rho \int dx dy$, $G_{AB} \equiv g^2 z_0^2 e^{2gz_*} \text{diag}(1, -1)$, $Y^A \equiv (x^0, z_*)$ and $gz_* \equiv \ln(z/z_0)$. This is the action of a point-particle moving in a (1+1)-dimensional space-time with metric G_{AB} and coordinates Y^A and it is straightforward to show that the Hamiltonian of this particle vanishes. The system is quantized by defining the commutation relations $[Y^A, p^B]_- = i\delta^{AB}$, where $p_A \equiv (p_{x^0}, p_{z_*})$ are the conjugate momenta, and the analogue of the Wheeler-DeWitt equation is

$$\left[\frac{\partial^2}{\partial z_*^2} - \frac{\partial^2}{\partial x_0^2} - (\rho' g z_0)^2 e^{2gz_*} \right] \Psi = 0. \tag{23}$$

Comparison with Eq. (18) therefore implies that this equation reduces to Eq. (2) after the coordinate transformation

$$u \equiv \frac{1}{2} b \rho' z_0 e^{g(z_* + x_0)}, \qquad v \equiv \frac{1}{2b} \rho' z_0 e^{g(z_* - x_0)}. \tag{24}$$

Hence, this model also admits the quantum and semi-classical solutions derived above.

In conclusion, we have shown how the general solutions to the Wheeler-DeWitt and Hamilton-Jacobi equations corresponding to a two-dimensional minisuperspace can be found analytically if the superpotential has the generic form given by Eq. (7). More generally, these equations can be solved whenever new coordinates $u = u(\sigma)$ and $v = v(\tau)$ can be found that satisfy Eq. (6).

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